Reference Answer of Lecture 2

Generalized Gronwall Inequality

Suppose that, $g(t), C(t), h(t) \in C([t_0, T])$ with $\mathrm{C(t)} \geq 0$ and $h(t) \geq 0$ .If

$$g(t) \leq C(t) + \int_{t_0}^{t} g(s)h(s)ds \quad \mathrm{t} \in [t_0, T]$$

Then we have

$$g(t) \leq C(t) + \int_{t_0}^{t} h(s)\mathrm{C(s)e}^{\int_s^t h(\tau)d\tau} ds \quad \mathrm{t} \in [t_0, T]$$

Proof: Let $u(t) = \int_{t_0}^{t} g(s)h(s)ds$ then $g(t) - C(t) \leq u(t) \quad \mathrm{t} \in [t_0, T]$

$$u'(t) = g(t)h(t) \leq (u(t) + C(t))h(t) = u(t)h(t) + C(t)h(t)$$

To solve this ODE,firstly we solve a general form $x'(t) = P(t)x(t) + Q(t)$ (*)

For ODE $x'(t) = P(t)x(t)$ we have $x(t) = ce^{\int_{t_0}^{t} P(s)ds}$

Then suppose $x(t) = c(t)e^{\int_{t_0}^{t} P(s)ds}$ is the general solution of ODE (*)
The first derivative of both sides:

$$x'(t) = c'(t)e^{\int_{t_0}^{t} P(s)ds} + c(t)e^{\int_{t_0}^{t} P(s)ds} P(t)$$

On the other sides

$$x'(t) = P(t)x(t) + Q(t) = P(t)c(t)e^{\int_{t_0}^{t} P(s)ds} + Q(t)$$

$$\therefore c'(t) = Q(t)e^{-\int_{t_0}^{t} P(s)ds} \Rightarrow x(t) = (c(t_0) + \int_{t_0}^{t} Q(s)e^{-\int_{t_0}^{s} P(\tau)d\tau} ds)e^{\int_{t_0}^{t} P(s)ds}$$

Here $P(t) = h(t), Q(t) = C(t)h(t)$

Then $u(t) = (c(t_0) + \int_{t_0}^{t} C(s)h(s)e^{-\int_{t_0}^{s} h(\tau)d\tau} ds)e^{\int_{t_0}^{t} h(s)ds}$

Because $u(t) = \int_{t_0}^{t} g(s)h(s)ds$,then $u(t_0) = 0$,while $u(t) = c(t)e^{\int_{t_0}^{t} h(s)ds} \Rightarrow c(t_0) = 0$

$$\therefore u(t) = (\int_{t_0}^{t} C(s)h(s)e^{-\int_{t_0}^{s} h(\tau)d\tau} ds) \; e^{\int_{t_0}^{t} h(s)ds} = \int_{t_0}^{t} C(s)h(s)e^{\int_s^t h(\tau)d\tau} ds$$

$$\therefore g(t) \leq C(t) + u(t) = C(t) + \int_{t_0}^{t} h(s)\mathrm{C(s)e}^{\int_s^t h(\tau)d\tau} ds \quad \mathrm{t} \in [t_0, T]$$